V. *Remarks on the probabilities of error in physical observations, and on the density of the earth, considered, especially with regard to the reduction of experiments on the pendulum. In a letter to Capt.* Henry Kater, *F. R. S. By* Thomas Young, *M. D. For. Sec. R. S.*

Read January 21, 1819.

My dear Sir,

The results of some of your late experiments on the pendulum having led me to reflect on the possible inequalities in the arrangement of gravitating matter within the earth's substance, as well as on the methods of appreciating the accuracy of a long series of observations in general, I have thought that it might be agreeable to you, to receive the conclusions which I have obtained from my investigations, in such a form as might serve either to accompany the report of your operations, or to be laid before the Royal Society as a distinct communication.

1. *On the estimation of the advantage of multiplied observations.*

It has been a favourite object of research and speculation, among the authors of the most modern refinements of mathematical analysis, to determine the laws, by which the probability of occurrences, and the accuracy of experimental results, may be reduced to a numerical form. It is indeed true, that this calculation has sometimes vainly endeavoured to substitute arithmetic for common sense, and at other times has exhibited an inclination to employ the doctrine of chances as a sort of auxiliary in the pursuit of a political object, not otherwise so easily attainable; but we must recollect, that at least as much good sense is required in applying our mathematics to objects of a moral nature, as would be sufficient to enable us to judge of all their relations without any mathematics at all: and that a wise government and a brave people may rely with much more confidence on the permanent sources of their prosperity, than the most expert calculators have any right to repose in the most ingenious combinations of accidental causes.

It is however an important, as well as an interesting study, to inquire in what manner the apparent constancy of many general results, which are obviously subject to great and numerous causes of diversity, may best be explained: and we shall soon discover that the combination of a multitude of independent sources of error, each liable to incessant fluctuation, has a natural tendency, derived from their multiplicity and independence, to diminish the aggregate variation of their joint effect; and that this consideration is sufficient to illustrate the occurrence, for example, of almost an equal number of dead letters every year in a general post office, and many other similar circumstances, which, to an unprepared mind, seem to wear the appearance of a kind of mysterious fatality, and which have sometimes been considered, even by those who have investigated the subject with more attention, as implying something approaching more nearly to constancy in the original causes of the events, than there is any just reason for inferring from them.

This statement may be rendered more intelligible by the simple case of supposing an equal large number of black and white balls to be thrown into a box, and 100 of them to be drawn out either at once or in succession. It may then be demonstrated, as will appear hereafter, from the number of ways in which the respective numbers of each kind of balls may happen to be drawn, that there is 1 chance in $12\frac{1}{2}$ that exactly 50 of each kind may be drawn, and an even chance that there will not be more than 53 of either, though it still remains barely possible that even 100 black balls or 100 white may be drawn in succession.

From a similar consideration of the number of combinations affording a given error, it will be easy to obtain the probable error of the mean of a number of observations of any kind; beginning first with the simple supposition of the certainty of an error of constant magnitude, but equally likely to fall on either side of the truth, and then deducing from this supposition the result of the more ordinary case of the greater probability of small errors than of larger ones. This liability to a constant error may be represented, by supposing a counter to have two faces, marked 0 and 2; the mean value of an infinite number of trials will then obviously be 1, and the constant error of each trial will be 1, whether positive or negative.

Now in a combination of n trials with such a counter, if we divide the sum of the results by n, the greatest possible error of the mean thus found will be 1; and the probability of any other given error will be expressed by the number of combination of the faces of n counters affording that error, divided by the whole number of combinations; that is, by the corresponding coefficient of the binomial $(1+1)^n$, divided by 2^n, the sum of the coefficients. The calculation therefore will stand thus:

	$n=2$			$n=3$				$n=4$					$n=6$				$n=8$				
Coefficients	1	2	1	1	3	3	1	1	4	6	4	1	1	6	15	20...	1	8	28	56	70...
Numbers thrown	0	2	4	0	2	4	6	0	2	4	6	8	0	2	4	6...	0	2	4	6	8...
Differences from n	2	0	2	3	1	1	3	4	2	0	2	4	6	4	2	0...	8	6	4	2	0...
Errors of the means	1	0	1	1	$\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{2}{3}$	$\frac{1}{3}$	0...	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0...
Sums of errors	$1+0+1=2$			$1+1+1+1=4$				$1+2+0+2+1=6$					$1+4+5+0...$				$1+6+14+14+0...$				
Mean errors	$\frac{2}{4}=\frac{1}{2}$			$\frac{4}{8}=\frac{1}{2}$				$\frac{6}{16}=\frac{3}{8}$					$\frac{20}{64}=\frac{5}{16}$				$\frac{70}{256}=\frac{35}{128}$				

It is easy to perceive that these coefficients must express the true numbers of the combinations, since they are formed by adding together the two adjacent members of the preceding series; thus when n is 3, 1 combination giving the number 0 and 3 the number 2, these two combinations, being again respectively combined with 2 and 0 of a fourth counter, give $1+3=4$, for the combinations affording the number 2 in the next series; while each succeeding series must continue to begin and end with unity, since there is only one combination that can afford either of the extremes.

In order to continue the calculation with greater convenience, we must find a general expression for the middle terms, 2, 6, 20, 70..., neglecting the odd values of n. The first, 2, is made up of $(1+1)$, the second, 6, is $2(2+1)$; 20 is $2(6+4)$ and $70=2(20+15)$: or $6=2(2.\frac{3}{2})$, $20=2(6.\frac{5}{3})$, $70=2(20.\frac{7}{4})$, whence the series may easily be continued at pleasure, multiplying always the preceding term by $\frac{6}{2}, \frac{10}{3}, \frac{14}{4}, \frac{18}{5}, \ldots$ We have also $6=16.\frac{3}{8}=2^4.\frac{3}{8}$, $20=2^6.\frac{5}{16}$, and $70=2^8.\frac{35}{128}$: consequently the terms of this series, divided by 2^n, will always express the mean errors already calculated. From this value of the middle term we may easily deduce that of the neighbouring terms by means of the original formula $n.\frac{n-1}{2}\ldots\frac{\frac{1}{2}n+1}{\frac{1}{2}n}.\frac{\frac{1}{2}n}{\frac{1}{2}n+1}.\frac{\frac{1}{2}n-1}{\frac{1}{2}n+2}\ldots$; the first factor less than unity being always $\frac{\frac{1}{2}n}{\frac{1}{2}n+1} = \frac{1}{1+\frac{2}{n}}$. The magnitude of the mean error is exhibited in the annexed table.

n	Mean error
2	.500000
4	.375000
6	.312500
8	.273437
10	.246094
12	.225586
14	.209473
16	.196381
18	.185471
20	.176196
30	.144466
40	.125363
50	.112271
60	.102574
70	.095022
80	.088924
90	.083868
100	.079586

The general expression for this series being $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \ldots \frac{n-1}{n}$, it is obvious that if we multiply it by $\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{n-2}{n-1}$, the product will be $\frac{1}{n}$, whatever the value of n may be: and when that value is large, the factors of these two expressions will approach so near to each other that they may be considered as equal; consequently the corresponding terms of either, taken between any two large values of n, will vary in the subduplicate ratio of n, since their product, which may be considered as the square of either, varies in the simple ratio of n, so that the mean error may ultimately be expressed by $\sqrt{\frac{1}{pn}}$. The value of p evidently approximates to that of the quadrant of a circle, of which the radius is unity: thus for $n = 10$ it is 1.6512, and for $n = 100$, 1.5788, instead of 1.5708; and the ultimate identity of these magnitudes has been demonstrated by EULER and others. (See Mr. HERSCHEL's Treatise on Series, in LACROIX, Engl. Ed. n. 410.)

The fraction thus found, multiplied by 2^n, gives the number of combinations expressed by the middle term, in which the error vanishes, when n is even: and the whole number of combinations being also 2^n, it is obvious that the fraction alone must express the probability of a result totally free from error. The neighbouring terms on each side, for $n = 100$, are .078025, .073524, and .066588, the sum of the 7 being .515860; and since this sum exceeds $\frac{1}{2}$, it is obviously more probable that the result of 100 trials will be found in some of these seven terms, than in any of the remaining 94, and that the mean error will not exceed $\frac{3}{50}$. When n is so large, that the terms concerned may be considered as nearly equal, the factors $\frac{\frac{1}{2}n}{\frac{1}{2}n+1}$, $\frac{\frac{1}{2}n-1}{\frac{1}{2}n+2}$..., may be expressed by $1 - \frac{2}{n}$, $1 - \frac{6}{n}$, $1 - \frac{10}{n}$..., and the terms themselves by 1, $1 - \frac{2}{n}$, $1 - \frac{8}{n}$, $1 - \frac{18}{n}$... the negative parts forming the series $\frac{2}{n}(1, 4, 9 \ldots)$ of which the sum, for q terms, is $\frac{2}{n}(\frac{1}{3}q^3 + \frac{1}{2}q^2 + \frac{1}{6}q)$ or ultimately $\frac{2}{3n}q^3$; consequently if we call the middle term e, we must determine q in such a manner as to have $e(2q - \frac{4}{3n}q^3) = \frac{1}{2} - e$, and $q(1 - \frac{2}{3n}q^2) = \frac{1}{4e} - \frac{1}{2}$; but e has been already found, in this case, $= \sqrt{\frac{1}{pn}}$, and neglecting at first the square of q, we have $q = \frac{1}{4}\sqrt{(pn)} - \frac{1}{2}$, and $q^2 = \frac{1}{16}pn$, whence $\frac{2}{3n}q^2 = \frac{1}{24}p$, and $1 - \frac{2}{3n}q^2 = .93455$; hence, for a second approximation, $.93455\,q = \frac{1}{4e} - \frac{1}{2}$, and $q = .2674\sqrt{(pn)} - .53$; and by continuing the operation we obtain $.9235\,q = \frac{1}{4e} - \frac{1}{2}$, and $q = .271\sqrt{(pn)} - .54$; consequently the probable error, being expressed by $\frac{2q}{n}$, will be $.542\sqrt{\frac{p}{n}} - \frac{1.08}{n} = \frac{.679}{\sqrt{n}} - \frac{1.08}{n}$. This formula, for $n = 100$, becomes .0571, and for $n = 10000$, $.00679 - .00011 = .00668$.

We must not, however, lose sight, in this calculation, of the original condition of liability to a certain constant error in each trial. For example, we may infer from it, that if we made 100 observations of the place of a luminary, each differing 1′ from the truth, but indifferently on either side of it, the error of the mean result would probably not exceed $\frac{3}{50} \cdot 1' = 3.6''$; and that in 1000 observations it would probably be reduced to about a second. Now although, in the methods of observing which we employ, the error is liable to considerable variations, yet it may be represented with sufficient accuracy, by the combination of two or more experiments in which the simpler law prevails. For example, the combination of two counters, such as have been considered, is equivalent to the effect of a die with four faces, or a tetraedron, marked 0, 2, 2, and 4, or with errors expressed by 1, 0, 0, and —1; the combination of three counters is represented by a die having eight faces, or an octaedron, with the errors 1, $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -1$; and the combination of four, by a solid of 16 sides, with the errors $1, 4 \times \frac{1}{2}, 6 \times 0, 4 \times -\frac{1}{2}, -1$. These distributions evidently resemble those which are generally found to take place in the results of our experiments; and it is of the less consequence to represent them with greater accuracy, since the minute steps, by which the scale of error varies, have no sensible effect on the result, especially when the number of observations is considerable. If, for example, instead of two trials with the tetraedron, having the errors 1, 0, 0, — 1, we made two trials with a solid of 21 faces, having the errors distributed equally from 1, .9, .8 .. to —1, the mean error of all the possible combinations would only vary from .375 to .349; and in a greater number of trials the errors would approach still nearer to equality.

Now in order to employ any of these suppositions for the purpose of calculation, it is only necessary to compute the corresponding mean error, and to make it equal to the actual mean error of a great number of observations. Thus, if we consider each observation as representing a binary combination of counters or constant errors, in which the mean error is $\frac{1}{2}$, and adding together the differences of the several results from the mean, and dividing by their numbers, we find the mean error of 100 observations $1'$, we must consider the original constant error as equal to $2'$, which is to be made the unit for 200 primitive combinations; and $\frac{.679}{\sqrt{200}} - \frac{1.08}{200} = .0426$; and the probable error of the mean will be $.0426 \times 120 = 5.1''$. For a quaternary combination, if the error, which amounts to $\frac{3}{8}$, be found $1'$, the unit will be $\frac{8}{3}'$, and for $n = 400$, we have $.03125 \times \frac{8}{3}' = 5.0''$. And if we set out with a large number m of combinations, the mean error being $\sqrt{\frac{1}{pm}} = e$, the unit will be $e\sqrt{(pm)} = 1$, and the probable error of nm trials being equal to this unit multiplied by $.542\sqrt{\frac{p}{nm}}$, neglecting the very small fraction $\frac{1.08}{nm}$, we have $.542\sqrt{\frac{p}{nm}}\, e\sqrt{(pm)} = .542\, p\sqrt{\frac{1}{n}}\, e = .8514\sqrt{\frac{1}{n}}\, e$: which, if e be $1'$, and $n = 100$, gives again $5.1''$. It appears therefore that the supposition, respecting the number of combinations representing the scale of error, scarcely makes a perceptible difference in the result, after the exclusion of the constant error: and that we may safely represent the probable error of the mean result of n observations, by the expression $.85\frac{e}{\sqrt{n}}$, e being the mean of all the actual errors.

We might obtain a conclusion nearly similar by considering the sum of the squares of the errors, amounting always to $n\,2^n$: but besides the greater labour of computing the sum of the squares of the errors of any series of observations, the method, strictly speaking, is somewhat less accurate, since the amount of this sum is affected in a slight degree by any error which may remain in the mean, while the simple sum of the errors is wholly exempted from this uncertainty. In other respects the results here obtained do not materially differ from those of LEGENDRE, BESSEL, GAUSS, and LAPLACE: but the mode of investigation appears to be more simple and intelligible.

It may therefore be inferred from these calculations, first, that the original conditions of the probability of different errors, though they materially affect the observations themselves, do not very greatly modify the nature of the conclusions respecting the accuracy of the mean result, because their effect is comprehended in the magnitude of the mean error from which those conclusions are deduced: and secondly, that the error of the mean, on account of this limitation, is never likely to be greater than six sevenths of the mean of all the errors, divided by the square root of the number of observations. But though it is perfectly true, that the probable error of the mean is always somewhat less than the mean error divided by the square root of the number of observations, provided that no constant causes of error have existed; it is still very seldom safe to rely on the total absence of such causes; especially as our means of detecting them must be limited by the accuracy of our observations, not assisted, in all instances, by the tendency to equal errors on either side of the truth; and when we are comparing a series of observations made with any one instrument, or even by any one observer, we can place so little reliance on the absence of some constant cause of error, much greater than the probable result of the accidental causes, that it would in general be deceiving ourselves even to enter into the calculation upon the principles here explained: and it is much to be apprehended, that for want of considering this necessary condition, the results of many elegant and refined investigations, relating to the probabilities of error, may in the end be found perfectly nugatory.

These are cases in which some little assistance may be derived from the doctrine of chances with respect to matters of literature and history: but even here it would be extremely easy to pervert this application in such a manner, as to make it subservient to the purpose of clothing fallacious reasoning in the garb of demonstrative evidence. Thus if we were investigating the relations of two languages to each other, with a view of determining how far they indicated a common origin from an older language, or an occasional intercourse between the two nations speaking them, it would be important to inquire, upon the supposition that the possible varieties of monosyllabic or very simple words must be limited by the extent of the alphabet to a certain number; and that these names were to be given promiscuously to the same number of things, what would be the chance that 1, 2, 3 or more of the names would be applied to the same things in two independent instances.

Now we shall find, upon consideration, that for n names and n things, the whole number of combinations, or rather permutations of the whole nomenclature would be $m = 1 . 2 .$

$3 \ldots n$; and that of these the number in which no one name agreed would be $a_n = m - a_1 - n . \frac{n-1}{2} . a_2 - n . \frac{n-1}{2} . \frac{n-2}{3} . a_3 \ldots - n . a_{n-1}$; each term expressing the number of agreements in $n, n-1, n-2 \ldots$ instances only, and being made up of all the combinations of so many out of n things, each occurring as many times as all the remaining ones can disagree. Hence we may easily obtain the successive values of a from each other, the first being obviously 1, as a single name can only be given in one way to a single thing, therefore,

$a_1 = 1$

$a_2 = 2 - 1 = 1$

$a_3 = 6 - 1 - 3 = 2$

$a_4 = 24 - 1 - 6 - 8 = 9$

$a_5 = 120 - 1 - 10 - 20 - 45 = 44$

$a_6 = 720 - 1 - 15 - 40 - 135 - 264 = 265$

$a_7 = 5040 - 1 - 21 - 70 - 315 - 924 - 1855 = 1854$

$a_8 = 40320 - 1 - 28 - 112 - 630 - 2464 - 7420 - 14832 = 14833$

$a_9 = 362880 - 1 - 36 - 168 - 1134 - 5544 - 22260 - 66744 - 133497 = 133496$

$a_{10} = 3628800 - 1 - 45 - 240 - 1890 - 11088 - 55650 - 222480 - 667485 - 1334960 = 1334961$

From this computation it may be inferred, that, for 10 names, the probabilities will stand thus:

No coincidence	.367880	One or more	.632120
One only	.367880	Two or more	.264240
Two only	.183941	Three or more	.080300
Three only	.061309	Four or more	.018991
Four only	.015336	Five or more	.003655
Five only	.003056	Six or more	.000599
Six only	.000521	Seven or more	.000078

Seven only	.000066	Eight or more	.000012
Eight only	.000012	Nine or Ten	.0000003

The same results may be still more readily obtained from the supposition that n is a very large number; for then, the probability of a want of coincidence for a single case being $\frac{n-1}{n}$, the probability for two trials will be $\left(\frac{n-1}{n}\right)^2$, and for the whole n, $\left(\frac{n-1}{n}\right)^n = \left(1-\frac{1}{n}\right)^n$: but the hyperbolical logarithm of $1-\frac{1}{n}$ being ultimately $-\frac{1}{n}$, that of $\left(1-\frac{1}{n}\right)^n$ will be -1, consequently the probability of no coincidence will be $\frac{1}{2.718282}$ $=.3678794$: and if n is increased by 1, each of these cases of no coincidence will afford 1 of a single coincidence: if by two, each will afford one of a double coincidence, but half of them will be duplicates; and if by three, the same number must be divided by 6, since all the combinations of three would be found six times repeated. We have therefore for

No coincidence	.3678794	One or more	$.6321206 = \frac{2}{3} -$
One only	.3678794	Two or more	$.2642412 = \frac{1}{4} +$
Two only	.1839397	Three or more	$.0803015 = \frac{1}{12} -$
Three only	.0613132	Four or more	$.0189883 = \frac{1}{53}$
Four only	.0153283	Five or more	$.0036600 = \frac{1}{273}$
Five only	.0030657	Six or more	$.0005943 = \frac{1}{1683}$
Six only	.0005109	Seven or more	$.0000834 = \frac{1}{12000}$
Seven only	.0000730	Eight or more	$.0000105 = \frac{1}{95200}$

It appears therefore that nothing whatever could be inferred with respect to the relation of two languages from the coincidence of the sense of any single word in both of them; and that the odds would only be 3 to 1 against the agreement of two words: but if three words appeared to be identical, it would be more than 10 to 1 that they must be derived in both cases from some parent language, or introduced in some other manner; six words would give near 1700 chances to 1, and 8 near 100,000: so that in these last cases the evidence would be little short of absolute certainty.

In the Biscayan, for example, or the ancient language of Spain, we find in the vocabulary accompanying the elegant essay of Baron W. VON HUMBOLDT, the words *berria*, new; *ora*, a dog; *guchi*, little; *oguia*, bread; *otsoa*, a wolf, whence the Spanish *onza;* and *zazpi*, or, as LACROZE writes it, *shashpi*, seven. Now in the ancient Egyptian, new is BERI; a dog, UHOR; little, KUDCHI; bread, OIK; a wolf, UONSH; and seven, SHASHF; and if we consider these words as sufficiently identical to admit of our calculating upon them, the chances will be more than a thousand to one, that, at some very remote period, an Egyptian colony established itself in Spain: for none of the languages of the neighbouring nations retain any traces of having been the medium through which these words have been conveyed.

On the other hand, if we adopted the opinions of a late learned antiquary, the probability would be still incomparably greater that Ireland was originally peopled from the same mother country: since he has collected more than 100 words which are certainly Egyptian, and which he considers as bearing the same sense in Irish; but the relation, which he has magnified into identity, appears in general to be that of a very faint resemblance: and this is precisely an instance of a case, in which it would be deceiving ourselves to attempt to reduce the matter to a calculation.

The mention of a single number, which is found to be indisputably correct, may sometimes afford a very strong evidence of the accuracy and veracity of a historian. If the number were indefinitely large, the probability that it could not have been suggested by accident would amount to an absolute certainty: but where it must naturally have been confined within certain moderate limits, the confirmation, though somewhat less absolute, may still be very strong. For example, if the subject were the number of persons collected together for transacting business, it would be a fair presumption that it must be between 2 or 3 and 100, and the chances must be about 100 to 1 that a person reporting it truly must have some good information; especially if it were not an integral number of tens or dozens, which may be considered as a species of units. Now it happens that there is a manuscript of DIODORUS SICULUS, which, in describing the funerals of the Egyptians, gives 42 for the number of persons who had to sit in judgment on the merits of the deceased: and in a multitude of ancient rolls of papyrus, lately found in Egypt, it may be observed, that 42 personages are delineated, and enumerated, as the judges assisting Osiris in a similar ceremony. It is therefore perfectly fair to conclude from this undeniable coincidence, that we might venture to bet 100 to 1, that the manuscript in question is in general more accurate than the others which have been collated; that DIODORUS SICULUS was a well informed and faithful historian; that the graphical representations and inscriptions in question do relate to some kind of judgment; and lastly, that the hieroglyphical numbers, found in the rolls of papyrus, have been truly interpreted.

2. *On the mean density of the earth.*

It has been observed by some philosophers, that the excess of the density of the central parts of the earth, above that of the superficial parts, is so great as to render it probable that the whole was once in a state of fluidity, since this is the only condition that would enable the heaviest substances to sink towards the centre. But before we admit this inference, we ought to inquire, how great would be the effect of pressure only in augmenting the mean density, as far as we can judge of the compressibility of the substances, which are the most likely to be abundant, throughout the internal parts of the structure.

Supposing the density at the distance x from the centre to be expressed by y, the fluxion dy will be jointly proportional to the thickness of the elementary stratum, or to its fluxion $-\mathrm{d}x$, to the actual density y, and to the attraction of the interior parts of the sphere, which varies as $\frac{\int yxx\mathrm{d}x}{xx}$; since the increment of pressure, and consequently that of density, depends on the combination of these three magnitudes: we have therefore $-n\mathrm{d}y = y\mathrm{d}x\frac{\int y\mathrm{d}xx}{xx}$; an equation which will readily afford us the value of y in a series of the form $1 + ax^2 + bx^4 + \ldots$.

In order to determine the coefficients, we must first find $\frac{\int yxx\mathrm{d}x}{xx} = \frac{1}{3}x + \frac{1}{5}ax^3 + \frac{1}{7}bx^5 + \ldots$, and multiplying this by $(1 + ax^2 + bx^4 + \ldots)\,\mathrm{d}x$, we obtain

$$-ny = -n - nax^2 - nbx^4 - ncx^6 - \ldots$$
$$= C + \frac{1}{2.3}x^2 + \left.\begin{matrix}\frac{1}{4.5}a \\ +\frac{1}{3.4}a\end{matrix}\right\} x^4 + \left.\begin{matrix}\frac{1}{6.7}b \\ +\frac{1}{5.6}a^2 \\ +\frac{1}{3.6}b\end{matrix}\right\} x^6 + \ldots$$

Hence, by comparing the corresponding terms, we obtain

$C =$	$-n$;		
$a =$	$-.1666667n^{-1}$	Logarithm,	9.2218487
$b =$	$.22222222n^{-2}$		8.3467875
$c =$	$-.00268960n^{-3}$		7.4296867
$d =$	$.000308154n^{-4}$		6.4887650
$e =$	$-.0000340743n^{-5}$		5.5324269
$f =$	$.00000367495n^{-6}$		4.5652514
$g =$	$-.000000389086n^{-7}$		3.5911459
[$h =$	$.0000000004062n^{-8}$		2.6087]
[$i =$	$-.00000000420n^{-9}$		1.6232]
[$k =$	$-.00000000043n^{-10}$		0.6335]

After the exact determination of the first seven coefficients, the next three are obtained with sufficient accuracy by means of the successive differences of the logarithms, compared with those of the natural numbers.

It happens very conveniently, that the conditions of the problem are such, as to afford a remarkable facility in deriving from this series another, which is much more convergent, and which gives us the hyperbolic logarithm of y; for since $-n\frac{\mathrm{d}y}{y} = \mathrm{d}x\frac{fyx\dot{x}\mathrm{d}x}{xx}$, and $\frac{fyx\mathrm{d}x}{xx} = \frac{1}{3}x + \frac{1}{5}ax^3 + \frac{1}{7}bx^5 + \ldots$, if we multiply this by $\mathrm{d}x$, and take the fluent, we shall have $\mathrm{HL}y = -\frac{1}{n}\left(\frac{1}{2.3}x^2 + \frac{1}{4.5}ax^4 + \frac{1}{6.7}bx^6 + \ldots\right)$.

We may determine the degree of compressibility corresponding to a given value of n, by comparing the equation $-n\frac{\mathrm{d}y}{y} = \mathrm{d}x\frac{fyx\mathrm{d}x}{xx}$, or $= \mathrm{d}xp$, with the properties of the modulus of elasticity M, which is the height of such a column of the given substance, that the increment of density y', occasioned by the additional weight of the increment x', is always to y, as x' to M, or $\frac{y'}{y} = \frac{x'}{M}$, whence $-\frac{dy}{y} = \frac{dx}{M}$; consequently in the present case we have $\frac{dxp}{n} = \frac{dx}{M}$; and $M = \frac{n}{p}$: and if we make $x = 1$ in the value of p, we shall obtain M in terms of the radius of the earth, considered as unity. When y is invariable, and n infinite, the density being uniform, p becomes $\frac{1}{3}$, and the mean density will always be expressed by $3p$, since the attractive force is simply as the mean density: and if we divide $3p$ by y, we shall have the relation of the mean density to the superficial density. The results of this calculation, for different values of n, are arranged in the table, which will be found sufficiently accurate for the purposes of the investigation, though not always correct to the last place of figures.

n	p	$M=\frac{n}{p}$	$3p$, mean density		y	$\frac{3p}{y}$, comp. den.
∞	.33333	∞	1.0000=1:	1.0000	1.000	1.000
1	.30290	3.301	.9087	1.1005	.855	1.065
$\frac{1}{2}$	.27735	1.803	.8320	1.2019	.738	1127
$\frac{1}{3}$	.25535	1.305	.7660	1.3054	.646	1.185
$\frac{1}{4}$	.23688	1.055	.7106	1.4071	.575	1.24
$\frac{1}{5}$	.22058	.907	.6617	1.5111	.510	1.30
$\frac{1}{6}$	.20616	.808	.6185	1.6168	.458	1.35
$\frac{1}{7}$	.194	.736	.582	1.72	.419	1.40
$\frac{1}{8}$	.183	.681	.549	1.82	.377	1.45
$\frac{1}{9}$	.172	.646	.516	1.94	.346	1.49
[$\frac{1}{10}$	.162	.617	.486	2.05	.320	1.52]
[$\frac{1}{11}$	.153	.594	.459	2.16	.298	1.55]
[$\frac{1}{12}$	.145	.575	.435	2.28	.28	1.57]
[$\frac{1}{20}$	.1	.5	.3	3.3	.17	1.8]

The reciprocals of the mean density are inserted, on account of the simplicity of the progression which they exhibit, being in the first instance precisely equal to $1 + \frac{1}{10n}$, and varying but slowly from this value.

Now if we suppose, with Mr. LAPLACE, the mean density of the earth to be to that of the superficial parts as 1.55 to 1, it appears from this table, that the height of the modulus of elasticity must be about .594; that is, more than 12 million feet, while the modulus of the hardest and most elastic substances, that have been examined, amounts only to about 10 million. It follows therefore, that the general law, of a compression proportionate to the pressure, is amply sufficient to explain the greater density of the internal parts of the earth; and the fact demonstrates, that this law, which is true for small pressures in all substances, and with regard to elastic fluids, in all circumstances, requires some little modification for solids and liquids, the resistance increasing somewhat faster than the density: for no mineral substance is sufficiently light and incompressible to afford a sphere of the magnitude of the earth, and of so small a specific gravity, without some such deviation from the general law. A sphere of water would be incomparably more dense, and one of air would exceed this in a still greater proportion: indeed, even the moon, if she is really perforated, as has sometimes been believed, and contains cavities of any considerable depth, would soon have absorbed into her substance the whole of her atmosphere, supposing that she ever had one. It may be objected, that the resistance of solids to actual compression may possibly be considerably greater than appears in our experiments, since we are not absolutely certain that they do not extend in a transverse direction, when we compress them in a longitudinal one, as is obviously the case with some soft elastic substances: but this objection is removed by the experiment on the sound of ice, which affords, either accurately or very nearly, the same resistance to compression as a portion of water confined in a strong vessel; and this it could not do, if the particles of ice were allowed to expand laterally under the operation of a compressing force.

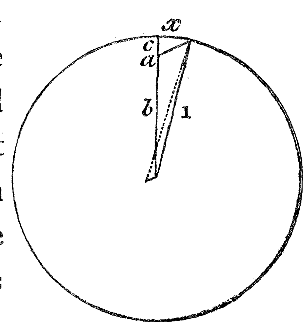
Mr. LAPLACE's conclusion, respecting the precise proportion of the densities, is indeed derived from another supposition respecting their variation, and would be somewhat modified by the adoption of this theory; it would not, however, be so materially altered, as by any means to invalidate the general inference. It would therefore be proper to revise the calculations derived from the lunar motions and the ellipticity of the earth, and to employ in them a variation of density somewhat resembling that which is here investigated. Indeed without reference to the effects of compressibility, it is obviously probable that the density of the earth should vary more considerably in a given depth towards the surface than near the centre, although the calculation, upon Mr. LAPLACE's more simple hypothesis, of a uniform variation, is much less intricate. It would however be justifiable, as a first approximation, to reject those terms of the series which would vanish if n and x were very small, and to make $y = 1 + ax^2$; and indeed this formula has in one respect an advantage over the series, as it seems to approach more nearly to the law of nature, in expressing a resistance somewhat greater towards the centre, where the density is most augmented: we have then, if the superficial density be to the mean as 1 to q, $q = \frac{1+\frac{3}{5}a}{1+a}$, whence $a = -\frac{q-1}{q-.6}$; and if $q =$ 1.55, $a = -.58$, affording an expression which is, in all probability, accurate enough for every astronomical purpose.

If the variation of density were supposed to proceed equably with the variation of quantity, it would obviously be as the square of the distance from the centre, and the density would be as $1 - ax^3$, the mean density being found at the surface of a sphere containing half as much as the whole earth; and this might be considered as the most natural hypothesis, if we disregarded the effects of compression: but the arithmetical progression of densities, from the centre to the surface, seems in every way improbable.

3. *On the irregularities of the earth's surface.*

A. If we suppose the plumb line to deviate from its general direction on account of the attraction of a circumscribed mass, situated at a moderate depth below the earth's surface, the distance of the two points of greatest deviation from each other will be to the depth of the attracting point as 2 to $\sqrt{2}$.

Let the magnitude of the additional mass be to that of the earth as a to 1, and let its distance from the centre be b; then supposing the earth a sphere, and its radius unity, and calling the angular distance of any point from the semidiameter passing through the mass x, the linear distance from the mass will be $\sqrt{(\text{f}^2x + (\text{ç}x - b)^2)} = \sqrt{(\text{f}^2x + \text{ç}^2x - 2b\text{ç}x + b^2)} = \sqrt{(1 + b^2 - 2b\text{ç}x)}$; consequently the disturbing attraction will be $\frac{a}{1+bb-2b\text{ç}x}$: but the sine of the angle subtended by the two



centres of attraction will be to their distance b as $\text{ſ}x$ to the oblique distance $\surd\,(1+b^2-2b\text{ç}x)$; it will therefore be expressed by $\frac{b\text{ſ}x}{\surd(1+bb-2b\text{ç}x)}$; and the sine of the very small angular deviation of the joint force from the radius will be to the line measuring the disturbing force as this last sine to the radius, the difference of the third side of the triangle from the radius being inconsiderable; consequently the deviation will be every where expressed by $\frac{ab\text{ſ}x}{(1+bb-2b\text{ç}x)^{\frac{3}{2}}}=d$. Now in order to find where this is greatest, we must make its fluxion vanish, and $0=\frac{\text{ç}x\,dx}{(1+bb-2b\text{ç}x)^{\frac{3}{2}}}-\frac{3}{2}\cdot\frac{\text{ſ}x\,2b\text{ſ}x\,dx}{(1+bb-2b\text{ç}x)^{\frac{5}{2}}}$, $\text{ç}x\,(1+b^2-2b\text{ç}x)=3b\text{ſ}^2x$, $3b\text{ç}^2x-2b\text{ç}^2x+(1+b^2)\,\text{ç}x=3b$, $\text{ç}^2x+\frac{1+bb}{b}\,\text{ç}x=3$, and $\text{ç}x=\surd\left(3+\left[\frac{1+bb}{2b}\right]^2\right)-\frac{1+bb}{2b}$; but, making $b=1-c$, $\frac{1+bb}{2b}$ becomes $\frac{1+1-2c+cc}{2b}=1+\frac{cc}{2b}$; and c being very small, $\text{ç}x$ will be $\surd\left(4+\frac{cc}{b}\right)-1-\frac{cc}{2b}=2+\frac{cc}{4b}-1-\frac{cc}{2b}=1-\frac{cc}{4b}$; whence $\text{ſ}x=\surd\left(1-\left[1-\frac{cc}{4b}\right]^2\right)=\surd\left(1-1+\frac{cc}{2b}\right)=\frac{c}{\surd(2b)}$. or simply $\surd\frac{1}{2}c$, and $c=\surd 2\text{ſ}x$.

B. The sine of the greatest deviation of the plumb line will amount to $d=.385\frac{a}{cc}$, a being the disturbing mass, and c its depth.

Since $\text{ç}x=1-\frac{cc}{4b}$, $2b\text{ç}x=2b-\frac{cc}{2}$, and $1+bb-2b+\frac{cc}{2}=(1-b)^2+\frac{cc}{2}=c^2+\frac{cc}{2}=\frac{3}{2}c^2$; and $ab\text{ſ}x$ becomes $\frac{abc}{\surd(2b)}$, whence $d=\frac{abc}{\surd\left(2b\cdot\frac{27}{8}\right)c^3}=\frac{2a\surd b}{\surd 27\,cc}=.385\frac{a\surd b}{cc}$, or simply $.385\frac{a}{cc}$; also $a=2.618c^2d$, and $c=\surd\left(.385\frac{a}{d}\right)$. If the density were doubled throughout the extent of a sphere touching the surface internally, the radius being c, we should have $a = c^3$ and $d = .385c$, and $c \doteq 2.6d$: but this is a much greater increase of density than is likely to exist on a large scale: so that c must probably in all cases be considerably greater than this.

C. The greatest elevation of the general surface above the sphere will be $\frac{a}{c}$, on the supposition that the mutual attraction of the elevated parts may safely be neglected.

The fluxion of the elevation is as the fluxion of the arc and as the deviation d conjointly; it will therefore be expressed by $\frac{ab\int x \mathrm{d}x}{(1+bb-2b\varsigma x)^{\frac{3}{2}}}$. Now the fluxion of $\frac{1}{\sqrt{(1+bb-2b\varsigma x)}}$ is $-\frac{1}{2}\frac{2b\int x \mathrm{d}x}{(1+bb-2b\varsigma x)^{\frac{3}{2}}}$, consequently the fluent of the elevation will be $\frac{-a}{\sqrt{(1+bb-2b\varsigma x)}}$: and while ςx varies from 1 to -1, this fluent will vary from $\frac{-a}{1-b}$ to $\frac{-a}{1+b}$, the difference being $a\left(\frac{1}{1-b} - \frac{1}{1+b}\right) = a\left(\frac{1}{c} - \frac{1}{2-c}\right) = a\left(\frac{2-c-c}{2c-cc}\right)$, or simply $\frac{a}{c}$, since c is an extremely small fraction. This quantity comprehends indeed the depression on the remoter side of the sphere, which would be required to supply matter for the elevation; but it is obvious that such a depression must be wholly inconsiderable.

D. The diminution of gravity to the centre at the highest point is $\frac{2a}{c}$, while the increase from the attraction of the disturbing mass is nearly $\frac{a}{cc}$, which is greater in the proportion that half the radius bears to c.

E. The increase of gravity, at the point of greatest deviation, is to the deviation itself, or its sine d, as $\sqrt{2}$ to 1.

For the deviation is the measure of the horizontal attraction of the disturbing mass, which is to its vertical attraction as fx to c, or as $\sqrt{\frac{1}{2}}$ to 1. Thus if d were 5″, or $\frac{5}{206265}$, the horizontal force would be $\frac{7.071}{206265} = \frac{1}{29170}$, and the acceleration of a pendulum $\frac{1}{58340}$ or 1.5″ of time in a day. It is true that a part of the deviation might depend on a defect of density as well as on an excess; but this defect could not amount to any great proportion of the whole, while the excess above the general density might easily be much more considerable, so that the acceleration of the pendulum could scarcely be *less* than a second in a day, if the greatest deviation of the plumb line were 5″; and if the deviation were 5″ at any other place, there would be a *greater* acceleration than a second at a point more or less remote from it.

F. If there were an excess of density on one side, and a deficiency on the other, so as to constitute virtually two centres of attraction and repulsion, and supposing their distances to be equal, and such as to produce the greatest deviation, if the excess of density were twice as great as the deficiency, a deviation of 5″ would correspond to an acceleration of half a second; if 3 times as great, to $\frac{3}{4}$; if 4 times, to $\frac{9}{10}$; and if five, to a second.

It may perhaps be considered as an omission in this calculation, that the attraction of the parts of the earth's surface, elevated by means of the irregular gravitation, has not been included in it. But it depends on the supposition that we may adopt respecting the cause and date of the irregularity, whether or no we ought to consider it as likely to have occasioned such a general elevation; and it does not appear that the result of the computation would very materially alter our conclusions, though it would be somewhat laborious to go through all its steps with precision. It would indeed be so much the more superfluous to insist on this minute accuracy, as variations so much more considerable in the form of the earth's surface are commonly neglected: for example, in the allowance made for the reduction of different heights to the level of the sea, which has usually been done without any consideration of the attraction of the elevated parts, interposed between the general surface and the place of observation. It is however obvious, that if we were raised on a sphere of earth a mile in diameter, its attraction would be about $\frac{1}{8000}$ of that of the whole globe, and instead of a reduction of $\frac{1}{2000}$ in the force of gravity, we should obtain only $\frac{3}{8000}$, or three fourths as much: nor is it at all probable that the attraction of any hill a mile in height would be so little as this, even supposing its density to be only two thirds of the mean density, of the earth: that of a hemispherical hill would be more than half as much more, or in the proportion of 1.586 to 1; and it may easily be shown, that the attraction of a large tract of table land considered as an extensive flat surface, a mile in thickness, would be three times as great as that of a sphere a mile in diameter: or about twice as great as that of such a sphere of the mean density of the earth: so that, for a place so situated, the allowance for elevation would be reduced to one half: and in almost any country that could be chosen for the experiment, it must remain less than three fourths of the whole correction, deduced immediately from the duplicate proportion of the distances from the earth's centre. Supposing the mean density of the earth 5.5, and that of the surface 2.5 only, the correction, for a tract of table land, will be reduced to $1-\frac{3}{4}\cdot\frac{2.5}{5.5}=\frac{29}{44}$, or $\frac{66}{100}$ of the whole.

4. EULER's *formula for the rolling pendulum.*

I beg leave to observe, in conclusion, with regard to Mr. LAPLACE's theorem for the length of the convertible pendulum rolling on equal cylinders, that its perfect accuracy may readily be inferred, without any limitation of the form of the pendulum, or of the magnitude of the cylinders, from the general and elegant investigation of EULER, which also affords us the proper correction for the arc of vibration. This admirable mathematician has demonstrated, in the sixth volume of the Nova Acta Petropolitana, for 1788, p. 145, that if we put k for the radius of gyration with respect to the centre of gravity, a for the distance of the centre of gravity from the centre of the cylinder, c for the radius of the cylinder, h^2 for $k^2 + (a - c)^2$, and b for the sine of half of any very small arc of semivibration, we shall have, for the time of a complete oscillation, $\frac{\pi h}{\sqrt{(2ag)}} + \frac{\pi bb(hh+4ac)}{4h\sqrt{(2ag)}}$, and ultimately, if $b = 0$, $\frac{\pi h}{\sqrt{(2ag)}}$ only, which, for a simple pendulum, of the length a, k and c both vanishing, becomes $\frac{\pi\sqrt{a}}{\sqrt{(2g)}}$, and for any other length l, $\frac{\pi\sqrt{l}}{\sqrt{(2g)}}$; consequently, making $\frac{\pi\sqrt{l}}{\sqrt{(2g)}} = \frac{\pi h}{\sqrt{(2ag)}}$, we have $\sqrt{l} = \frac{h}{\sqrt{a}}$, and $al = hh = k^2 + a^2 - 2ac + c^2$. Now if we find another value of a, which will fulfil the conditions of the equation, all the other quantities concerned remaining unaltered, and add the two values together, we shall have the distance of the centres of the two cylinders corresponding to the length l of the equivalent pendulum; but since $a^2 - (l + 2c)\,a = -k^2 - c^2$, we have $a - \frac{1}{2}l - c = \pm\sqrt{\ldots}$, and $a = \frac{1}{2}l + c \pm \sqrt{\ldots}$, so that the sum of the two values of *a* must be $l + 2c$, that is, the distance of the centres of the cylinders must exceed the length *l* by twice the radius, and *l* must be precisely equal to the distance of their surfaces.

Believe me, dear Sir,

very sincerely your's,

THOMAS YOUNG.

Welbeck Street, 29 Dec. 1818.

5. *Corrections for Refraction.*

1. A simple and convenient method of calculating the precise magnitude of the atmospherical refraction, in the neighbourhood of the horizon, has generally been considered as almost unattainable; and Dr. BRINKLEY has even been disposed to assert the "impossibility of investigating an exact formula," notwithstanding the "striking specimens of mathematical skill, which," as he justly observes, "have been exhibited in the inquiry." We shall find, however, that the principal difficulties may be evaded, if not overcome, by some very easy expedients.

2. The distance from the centre of the earth being represented by x, and the weight of the superincumbent column by y, the actual density may be called z, and the element of y will vary as the element of x and as the density conjointly; consequently $dy = -mzdx$; the constant quantity m being the reciprocal of the modulus of elasticity. The refractive density may be called $1+pz$, p being a very small fraction; and it is easy to see that the perpendicular u, falling on the direction of the light, will always vary inversely as the refractive density, since that perpendicular continually represents the sines of the consecutive angles, belonging to each of the concentric surfaces, at which the refraction may be supposed to take place. (Nat. Phil. II. p. 81:) and $u = \frac{s}{1+pz}$, s being a constant quantity. The angular refraction at each point will obviously be directly as the elementary change of this perpendicular, and inversely as the distance v from the point of incidence; whence the fluxion of the refraction will be $\frac{du}{v} = dr$, as is already well known.

3. For the fluent of this expression, which cannot be directly integrated, we may obtain a converging series by means of the TAYLORIAN theorem; but we must make the fluxion of the refraction constant, and that of the density variable; so that the equation will be $u = \frac{d\upsilon}{dr}\cdot r + \frac{d^2\upsilon}{dr^2}\cdot\frac{r^2}{2} + \frac{d^3\upsilon}{dr^3}\cdot\frac{r^3}{2.3} + \ldots$, υ being the initial value of u, when $r = 0$. Now the whole variation, of which u is capable, while z decreases from 1 to 0, extends from $\frac{s}{1+p}$ to s; or, since p is very small, from $s - ps$ to s; and dr being $= \frac{du}{v}$, we have the equation $ps = vr + \frac{dv}{dr}\cdot\frac{r^2}{2} + \ldots$ But $v = \sqrt{(x^2 - u^2)}$, $dv = \frac{xdx - udu}{v}$, and $\frac{dv}{dr} = \frac{x}{v}\cdot\frac{dx}{dr} - u$; and dx being $= -\frac{dy}{mz}$, and $du = -psdz$, $\frac{dx}{dr} = \frac{v}{mpsz}\cdot\frac{dy}{dz}$.

4. We must now determine the value of the density z, which, when the temperature is uniform, becomes simply $= y$; but for which we must find some other function of y, including the variation of temperature; and we may adopt, for this purpose, the hypothesis lately advanced by Professor LESLIE, in the article Climate of the Encyclopædia Britannica, and suppose the density to be augmented, by the effect of cold, in the proportion of 1 to $1 + n\left(\frac{1}{z} - z\right)$, n being somewhat less than $\frac{1}{10}$; and since the density is as the pressure and the comparative specific gravity conjointly, we have $z = y\left(1 + n\left[\frac{1}{z} - z\right]\right)$, $\frac{z}{y} = 1 + \frac{n}{z} - nz$, $d\frac{z}{y} = \frac{dz}{y} - \frac{zdy}{yy} = -\frac{ndz}{zz} - ndz$, and $\frac{dy}{dz} = \frac{y}{z} + \frac{nyy}{z^3} + \frac{nyy}{z}$; consequently $\frac{dx}{dr} = \frac{v}{mpsz}\left(\frac{y}{z} + \frac{nyy}{z^3} + \frac{nyy}{z}\right)$ and $\frac{dv}{dr} = \frac{xy}{mpszz} + \frac{nxyy}{mpszz} + \frac{nxyy}{mpsz^4} - u$. We may proceed to take the next fluxion with respect to y, z, and v, the variations of u and x being comparatively inconsiderable: so that if we call $\frac{dv}{dr} = X + Y + Z - s$, its fluxion will be $X\left(\frac{dy}{ydr} - \frac{2dz}{zdr}\right) + Y\left(\frac{2dy}{ydr} - \frac{2dz}{zdr}\right) + Z\left(\frac{2dy}{ydr} - \frac{4dz}{zdr}\right)$: but since $\frac{dy}{ydr} = \frac{-v}{psz} - \frac{2nvy}{psz^3} - \frac{2nvy}{psz}$, and $\frac{dz}{zdr} = \frac{-v}{psz}$, we have $\frac{d^2v}{dr^2} = X\left(\frac{v}{psz} - \frac{2nvy}{psz^3} - \frac{2nvy}{psz}\right) + Y\left(-\frac{2nvy}{psz^3} - \frac{2nvy}{psz}\right) + Z\left(\frac{2v}{psz} - \frac{4nvy}{psz^3} - \frac{4nvy}{psz}\right) = \frac{vx}{mp^2s^2}\left(\frac{y}{z^3} - \text{“}\frac{2ny^2}{z^5}\text{”} - \frac{2ny^2}{z^3} - \frac{2n^2y^3}{z^5} - \frac{2n^2y^3}{z^3} + \right.$

$\frac{\text{“}2ny^2\text{”}}{z^5} - \frac{4n^2y^2}{z^7} - \frac{4n^2y^2}{z^5}\Big)$, or, initially $= \frac{v}{mp^2s^2}(1-2n-2n^2-2n^2-4n^2-4n^2) = \frac{1-2n-12nn}{mp^2s^2}v$. In the next place, calling this fluxion H (K—L—M—N—P—Q) we obtain, for the fourth, $H(K-L-M-N-P-Q)\frac{dv}{vdr} + HK\left(\frac{dy}{ydr}-\frac{3dz}{zdr}\right) - HL\left(\frac{2dy}{ydr}-\frac{3dz}{zdr}\right) - HM\left(\frac{3dy}{ydr}-\frac{5dz}{zdr}\right) - HN\left(\frac{3dy}{ydr}-\frac{3dz}{zdr}\right) - HP\left(\frac{2dy}{ydr}-\frac{7dz}{zdr}\right) - HQ\left(\frac{2dy}{ydr}-\frac{5dz}{zdr}\right) = H(K-L-M-N-P-Q)\left(\frac{x}{mps}\left(\frac{y}{zz}+\frac{nyy}{zz}+\frac{nyy}{z^4}\right)-u\right) + HK\frac{v}{ps}\left(\frac{2}{z}-\frac{2ny}{z^3}-\frac{2ny}{z}\right) - HL\frac{v}{ps}\left(\frac{1}{z}-\frac{4ny}{z^3}-\frac{4ny}{z}\right) - HM\frac{v}{ps}\left(\frac{2}{z}-\frac{6ny}{z^3}-\frac{6ny}{z}\right) - HN\frac{v}{ps}\left(-\frac{6ny}{z^3}-\frac{6ny}{z}\right) - HP\frac{v}{ps}\left(\frac{5}{z}-\frac{4ny}{z^3}-\frac{4ny}{z}\right) - HQ\frac{v}{ps}\left(\frac{3}{z}-\frac{4ny}{z^3}-\frac{4ny}{z}\right) =$ $\frac{x^2}{m^2p^3s^3}\left(\frac{y^2}{z^5}-\frac{2ny^3}{z^5}-\frac{2n^2y^4}{z^7}-\frac{2n^2y^4}{z^5}-\frac{4n^2y^3}{z^9}-\frac{4n^2y^3}{z^7}+\frac{ny^3}{z^5}-\frac{2n^2y^4}{z^5}-\frac{2n^3y^5}{z^7}-\frac{2n^3y^5}{z^5}-\frac{4n^3y^4}{z^9}-\frac{4n^3y^4}{z^7}+\frac{ny^3}{z^7}-\frac{2n^2y^4}{z^7}-\frac{2n^3y^5}{z^9}-\frac{2n^3y^5}{z^7}-\frac{4n^3y^4}{z^{11}}-\frac{4n^3y^4}{z^9}\right) - \frac{x}{mp^2s}\left(\frac{y}{z^3}-\frac{2ny^2}{z^3}-\frac{2n^2y^3}{z^5}-\frac{2n^2y^3}{z^3}-\frac{4n^2y^2}{z^7}-\frac{4n^2y^2}{z^5}\right) + \frac{v^2x}{mp^3s^3}\left(\frac{2y}{z^4}-\frac{2ny^2}{z^6}-\frac{2ny^2}{z^4}-\frac{2ny^2}{z^4}+\frac{8n^2y^2}{z^6}+\frac{8n^2y^2}{z^4}-\frac{4n^2y^3}{z^6}+\frac{12n^3y^4}{z^8}+\frac{12n^3y^4}{z^6}+\frac{12n^3y^4}{z^6}+\frac{12n^3y^4}{z^4}-\frac{20n^2y^2}{z^8}+\frac{16n^3y^3}{z^{10}}+\frac{16n^3y^3}{z^8}-\frac{12n^2y^2}{z^6}+\frac{16n^3y^3}{z^8}+\frac{16n^3y^3}{z^6}\right)$. It will be unnecessary to continue the whole series any further; but it will be satisfactory to obtain that part of the sixth term, which is independent of v; and for this purpose we must take the fluxion of the first part with respect to y and z, and then with respect to v; and that of the second twice with respect to v only; and it will be sufficient in this case to employ the initial values of $\frac{dy}{dr}$, $\frac{dz}{dr}$, and $\frac{dv}{dr}$, which are $\frac{-v(1+4n)}{ps}$, $\frac{-v}{ps}$, and $\frac{1+2n}{mps}-s$; and calling $1+4n=k$, the part required will be $\Big(\frac{1}{m^2p^4s^4}(-2k+5+6kn-10n+8kn^2-14n^2+8kn^2-10n^2+12kn^2-36n^2+12kn^2-28n^2-3kn+5n+8kn^2-10n^2+10kn^3-14n^3+10kn^3-10n^3+16kn^3-36n^3+16kn^3-28n^3-3kn+7n+8kn^2-14n^2+10kn^3-18n^3+10kn^3-14n^3+16kn^3-44n^3+16kn^3-36n^3) - \frac{1}{mp^3s^2}[-k+3+4kn-6n+6kn^2-10n^2+6kn^2-6n^2+8kn^2-28n^2+8kn^2-20n^2]\Big)\left(\frac{1+2n}{mps}-s\right) + 2\left(\frac{1+2n}{mps}-s\right)^2\frac{1}{mp^3s^3}(2-2n-2n-2n+8n^2+8n^2-4n^2+12n^3+12n^3+12n^3+12n^3-20n^2+16n^3+16n^3-12n^2+16n^3+16n^3) =$ $\left(\frac{1}{m^2p^4s^4}(3-6n-56n^2+128n^3+416n^4) - \frac{1}{mp^3s^2}[2-6n-20n^2+112n^3]\right)\left(\frac{1+2n}{mps}-s\right) + 2\left(\frac{1+2n}{mps}-s\right)^2\frac{1}{mp^3s^3}(2-6n-20n^2+112n^3) + \ldots$

6. The whole equation becomes therefore ultimately $ps = vr + \left(\frac{1+2n}{2mps}-\frac{s}{2}\right)r^2 + \frac{1-2n-12nn}{6mp^2s^2}vr^3 + \left(\frac{1-16n^2-24n^3}{24m^2p^3s^3} - \frac{1-2n-12nn}{24mp^2s} + \frac{2-6n-20n^2+112n^3}{24mp^3s^3}v^2\right)r^4 + {*}{*} + \left(\left[\frac{3-6n-56n^2+128n^3+416n^4}{720m^2p^4s^4} - \frac{2-6n-20n^2+112n^3}{720mp^3s^2}\right]\left(\frac{1+2n}{mps}-s\right) + 2\left(\frac{1+2n}{mps}-s\right)^2\frac{2-6n-20n^2+112n^3}{720mp^3s^3} + \ldots\right)r^6 + \ldots$ We also obtain, for finding, on this hypothesis, the height x, corresponding to the pressure y and the density z, the expression $mx - m = 1 - \frac{y}{z} + \frac{n}{q}\,\mathrm{hl}\,\frac{2zz+qy(1-z)}{2zz-qy(1-z)}$; y being $=\frac{zz}{z+n-nzz}$, and $q^2 = 1+4n^2$. But the utility of the TAYLORIAN theorem, thus applied, in obtaining a series, is not confined to Professor LESLIE's hypothesis: it is equally well adapted to that of LAPLACE, or to any other admissible supposition respecting the distribution of temperatures: and we may therefore employ it in examining the comparative accuracy of the results of these different hypotheses.

7. Now if we take for n the value $\frac{45}{500} = .09$, corresponding to the multiplier 45, employed by Mr. LESLIE, the refractions in the immediate neighbourhood of the horizon will become too great by about 1′; a difference by far too considerable to be attributed to the errors of observation only; and we must infer, that the law of temperature, obtained from the height of the line of congelation, is not correctly true, if applied to elevations remote from the earth's suface. Professor BESSEL's approximation is also found to make the horizontal refraction too great. Mr. LAPLACE's formula, which affords a very correct determination of the refraction, is said to agree sufficiently well with direct observation also; but in fact this formula gives a depression considerably greater than was observed by GAY LUSSAC, in the only case which is adduced in its support; and the progressive depression follows a law which appears to be opposite to that of nature, the temperature varying less rapidly at greater than at smaller heights, while the observations of HUMBOLDT and others seem to prove that in nature they vary more rapidly. Notwithstanding, therefore, the ingenuity, and even utility of Mr. LAPLACE's formula, it can only be considered as an optical hypothesis, and we are equally at liberty to employ any other hypothesis which represents the results with equal accuracy; or even to correct our formulas by comparison with astronomical observations only, without assigning the precise law of temperature implied by them. The theory will however afford us some general indications for this purpose; showing, for example, that the coefficient of the second term cannot be smaller than $\frac{1}{2mps} - \frac{1}{2}s$, whatever positive value we may attribute to n; and if we adjust the second and fourth coefficients, so as to represent the refractions near the zenith and at the horizon, without regarding the value of the subsequent terms, we shall obtain the third, by dividing the fourth by half of the second; since that part of the fourth coefficient, which occurs in the case of horizontal refraction, is always derived from the third by taking the fluxion with respect to v only, and is therefore found by multiplying the third by $\frac{dv}{4vdr}$, whatever the relations of the other quantities concerned may be.

8. On every supposition, the coefficient of the first term must be $\frac{v}{s}$, and that of the second must not greatly differ from $\frac{3}{s} - \frac{s}{2}$. The third coefficient, on the hypothesis of a law analogous to Mr. LESLIE's, will be $1500\frac{v}{s}$; if we suppose the temperature to vary more uniformly, and make $z = y(1 + tx - t)$, the number will become 1900; or, taking $z = yx^t$, 2200, m being 766, and t 176: and Mr. LAPLACE's formula will of course give a value still larger. In fact the result of observation is represented with sufficient accuracy by the equation $.0002825 = v\frac{r}{s} + (2.5 + .5v^2)\frac{r^2}{s^2} + 3400\,v\frac{r^3}{s} + 3400\,(1.25 + .25\,v^2)\frac{r^4}{s^4}$, the barometer standing at 30 inches, and the thermometer at 50°: and this formula appears to be at least as accurate as the French tables. We have, for example:

Altitude	Refr.	Conn. d. T.	Altitude	Refr.	Conn. d. T.
° ′	′ ″	′ ″	° ′	′ ″	′ ′
0 0	33 52	33 52	20 0	2 39	2 40
5 0	9 57	9 56	30 0	1 41	1 41
10 0	5 21	5 21	45 0	58.15	58.3

The difference is somewhat greater a few degrees above the horizon; thus at 2° 17′ 50″, this formula makes the refraction 17′ 16″, the French tables 17′ 4″, BRADLEY's 17′ 30″, and Dr. BRINKLEY's observations reduced, 17′ 9″: but in such cases we can scarcely expect a greater degree of accuracy.

9. The terrestrial refraction may be most easily determined by an immediate comparison with the angle subtended at the earth's centre, the fluxion of which is $\frac{udx}{vx}$, and $\frac{udx}{vxdr}$ is initially the first part of the coefficient of the second term of the series already obtained, and is equal to 6; so that this angle, while it remains small, is six times the refraction: commonly, however, the refraction in the neighbourhood of the earth's surface is somewhat less than in this proportion.

10. The effects of barometrical and thermometrical changes may be deduced from the fluxion of the equation, if we make m, p, and n, or rather t, vary: and for this purpose it will be convenient to employ the form

$ps = vr + \left(\frac{1}{2(m-t)ps} + \frac{s}{2}\right) r^2$, the value of the fraction, if we neglect the subsequent terms, becoming 3.41; and this expression is sufficiently accurate for calculating the whole refraction, except for altitudes of a few degrees. Now the fluxion of $p = v\frac{r}{s} + \left(\frac{1}{2(m-t)p} - \frac{ss}{2}\right)\frac{rr}{ss}$, which we may call $p = v\frac{r}{s} + \left(\frac{1}{w} - \frac{ss}{2}\right)\frac{rr}{ss}$, is $dp = \left(\frac{v}{s} + \left(\frac{1}{w} - \frac{ss}{2}\right)\frac{2r}{ss}\right) dr - \frac{rr}{ssw}\left(\frac{dm - dt}{m - t} + \frac{dp}{p}\right)$, the coefficient of dr being equal to $\frac{2p}{r} - \frac{v}{s}$; and $\left(2p - \frac{rv}{s}\right)\frac{dr}{r} = \left(p + \frac{rr}{ssw}\right)\frac{dp}{p} + \frac{rr}{ssw}\left(\frac{dm - dt}{m - t}\right)$; $\frac{1}{w}$ being 3.41, and $m - t$, on this supposition, 519. The proportional variation of p, or $\frac{dp}{p}$, will be $\frac{1}{500}$ for every degree that the thermometer varies from 50°; and $\frac{dm}{m}$ being also $\frac{1}{500}$, $\frac{dm}{m-t}$ will be $\frac{766}{519 \times 500} = .003$. The variation of t can only be determined from conjecture; but supposing the alteration of temperature to cease at the height of about 4 miles, it must increase, with every degree that the thermometer rises at the earth's surface, about $\frac{1}{120}$, and $\frac{dt}{t}$ being $\frac{1}{120}$, $\frac{dt}{m-t}$ will be $\frac{247}{519 \times 120} = .004$. The alterations of the barometer will affect p only, $\frac{dp}{p}$ being $\frac{1}{30}$ for every inch above or below 30. It is evident, since $m = \frac{3958 \times 5280 \times 12}{13.57\, bd}$, b being the height of the barometer, and d the bulk of air compared to that of water, that m must diminish, as well as p, when the temperature increases; and the correction for t being subtractive, the three variations will co-operate in their effects; but the proportion will be somewhat different from that of the simple densities. If we preferred the expression derived from Professor LESLIE's hypothesis, we should merely have to substitute $\frac{2dn}{1 + 2n}$ for $\frac{dt}{m - t}$, and the variation depending on the law of temperature would become about $\frac{2}{3}$ as great. It must however be limited to such changes as affect the lower regions of the atmosphere only, its "argument" being the deviation from the mean temperature of the latitude; but even in this form it cannot be satisfactorily applied to the observations at present existing; although it appears to be amply sufficient to explain the irregularities of terrestrial refraction, as well as the uncommon increase of horizontal refraction in very cold countries: and we may even derive from all these considerations a correction of at least half a second, or perhaps of a whole second, for the sun's altitude at the winter solstice, tending to remove the discordance, which has so often been found, in the results of some of the most accurate observations of the obliquity of the ecliptic.

T. Y.